

τ -RIGID MODULES OVER AUSLANDER ALGEBRAS

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ABSTRACT. We give a characterization of τ -rigid modules over Auslander algebras in terms of projective dimension of modules. Moreover, we show that for an Auslander algebra Λ admitting finite number of non-isomorphic basic tilting Λ -modules and tilting Λ^{op} -modules, if all indecomposable τ -rigid Λ -modules of projective dimension 2 are of grade 2, then Λ is τ -tilting finite.

1. INTRODUCTION

Recently Adachi, Iyama and Reiten [AIR] introduced τ -tilting theory to generalize the classical tilting theory in terms of mutations. τ -tilting theory is close to the silting theory introduced by [AiI] and the cluster tilting theory in the sense of [KR, IY, BMRRT].

Note that τ -tilting theory depends on τ -rigid modules. So it is very interesting to find all τ -rigid modules for a given algebra. There are some works on this topic (See [A1, A2, AAC, IJY, IRR, J, M, HuZh, AnMV, W, Z] and so on). In particular, Iyama and Zhang [IZ] classified all the support τ -tilting modules and indecomposable τ -rigid modules for the Auslander algebra Γ of $K[x]/(x^n)$. They showed that the number of non-isomorphic basic support τ -tilting Γ -modules is exactly $(n+1)!$. For an arbitrary Auslander algebra Λ , little is known on τ -rigid Λ -modules. So a natural question is:

Question 1.1. *How to judge τ -rigid modules over an arbitrary Auslander algebra?*

Our first goal in this paper is to give a partial answer to this question. Throughout this paper all algebras are finite-dimensional algebras over a field K and all modules are finitely generated right modules.

For an algebra Λ , denote by $(-)^*$ the functor $\text{Hom}_\Lambda(-, \Lambda)$. For a Λ -module M , denote by $\text{pd}_\Lambda M$ (resp. $\text{id}_\Lambda M$) the projective dimension (resp. injective dimension) of M . Denote by $\text{grade} M$ the grade of M . Then we have the following theorem.

Theorem 1.2. *(Theorems 3.3 and 3.10, Corollary 3.7) Let Λ be an Auslander algebra and M a Λ -module. Then we have the following:*

- (1) *Every simple module S is τ -rigid.*
- (2) *If $\text{pd}_\Lambda M = 1$, then M is (τ) -rigid if and only if $\text{Ext}_\Lambda^2(N, M) = 0$, where $N = M^{**}/M$.*
- (3) *If $\text{grade} M = 2$, then M is τ -rigid if and only if $\text{Tr} M$ is τ -rigid with $\text{pd}_\Lambda \text{Tr} M = 1$.*
- (4) *If Λ admits a unique simple module S with $\text{pd}_\Lambda S = 2$, then*
 - (a) *Every indecomposable module M with $\text{pd}_\Lambda M = 1$ is (τ) -rigid.*
 - (b) *All indecomposable τ -rigid Λ -modules N with $\text{pd}_\Lambda N = 2$ are of grade 2.*

On the other hand, Demonet, Iyama and Jasso gave a general description of algebras with finite number of support τ -tilting modules in [DIJ] where they call the algebras τ -tilting finite algebras. It is clear that an algebra Λ is τ -tilting finite if and only if so is its opposite algebra Λ^{op} . We should remark that an algebra is τ -tilting finite implies that there are finite number of non-isomorphic basic tilting Λ -modules and tilting Λ^{op} -modules. It is natural to consider the following question.

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Question 1.3. *When is an algebra admitting finite number of basic tilting Λ -modules and tilting Λ^{op} -modules τ -tilting finite?*

It is obvious that algebras of finite representation type are both tilting-finite and τ -tilting finite. However, we need a non-trivial case. Our second goal of this paper is to give a more general answer to this question whenever Λ is an Auslander algebra. We prove the following theorem in which the algebra is not necessary to be an Auslander algebra.

Theorem 1.4. *(Theorem 3.8) Let Λ be an algebra of global dimension 2 admitting finite number of basic tilting Λ -modules and tilting Λ^{op} -modules. If all indecomposable τ -rigid modules with projective dimension 2 are of grade 2, then Λ is τ -tilting finite.*

The paper is organized as follows:

In Section 2, we recall some preliminaries. In Section 3, we prove the main results and give some examples to show the main results.

Throughout this paper, all algebras Λ are basic connected finite dimensional algebras over an algebraic closed field K and all Λ -modules are finitely generated right modules. Denote by $\text{mod } \Lambda$ the category of finitely generated right Λ -modules. For $M \in \text{mod } \Lambda$, denote by $\text{add } M$ the subcategory of direct summands of finite direct sum of M . We use $\text{Tr } M$ to denote the Auslander transpose of M . Denote by τ the AR -translation and denote by $|M|$ the number of non-isomorphic indecomposable direct summands of M .

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2. PRELIMINARIES

In this section we recall some basic preliminaries for later use. For an algebra Λ , denote by $\text{gl.dim } \Lambda$ the global dimension of Λ . We begin with the definition of Auslander algebras.

Definition 2.1. An algebra R is called an *Auslander algebra* if $\text{gl.dim } R \leq 2$ and $I_i(R)$ is projective for $i = 0, 1$, where $I_i(R)$ is the $(i + 1)$ -th term in a minimal injective resolution of R .

Let R be a representation-finite algebra and A an additive generator of $\text{mod } R$. Auslander proved that there is a one to one correspondence between representation-finite algebras and Auslander algebras via $R \mapsto \text{End}_R(A)$. In this case, we call $\text{End}_R(A)$ the *Auslander algebra of R* . Furthermore, for $X \in \text{mod } R$ we denote by $P_X = \text{Hom}_R(A, X)$ and $S_X = P_X / \text{rad } P_X$. The following statement [AuRS] is essential in the proof of the main result.

Proposition 2.2. *Let X be an indecomposable R -module. Then*

- (1) $\text{pd}_\Lambda S_X \leq 1$ if and only if X is projective, and $0 \rightarrow P_{\text{rad } X} \rightarrow P_X \rightarrow S_X \rightarrow 0$ is a minimal projective resolution of S_X .
- (2) $\text{pd}_\Lambda S_X = 2$ if and only if X is not projective, and the almost split sequence $0 \rightarrow \tau X \rightarrow E \rightarrow X \rightarrow 0$ gives a minimal projective resolution $0 \rightarrow P_{\tau X} \rightarrow P_E \rightarrow P_X \rightarrow S_X \rightarrow 0$ of S_X .

For a positive integer k , an algebra Λ is called Auslander's k -Gorenstein if $\text{pd}_\Lambda I_j(\Lambda) \leq j$ for $0 \leq j \leq k - 1$. For a Λ -module M and a positive integer i , we call $\text{grade } M \geq i$ if $\text{Ext}_\Lambda^j(M, \Lambda) = 0$ for $0 \leq j \leq i - 1$. We need the following result.

Lemma 2.3. *Let Λ be an Auslander algebra and $T \in \text{mod } \Lambda$. For $j = 1, 2$,*

- (1) *The subcategory $\{M | \text{grade } M \geq j\}$ is closed under submodules and factor modules.*
- (2) *Every simple Λ -module S is either of grade 0 or of grade 2.*
- (3) $\text{grade Ext}_\Lambda^j(T, \Lambda) \geq 2$.

(4) *The projective dimension of any composition factor of $\text{Ext}_\Lambda^2(T, \Lambda)$ is 2.*

Proof. (1) is a straight result of [I, Proposition 2.4].

(2) follows from the fact $\text{Ext}_\Lambda^i(S, \Lambda) \simeq \text{Hom}_\Lambda(S, I_i(\Lambda))$ and Λ is an Auslander algebra.

(3) By the definition of Auslander algebra, Λ is Auslander's 2-Gorenstein. Then by [FGR] Λ is Auslander's k -Gorenstein if and only if for each submodule X of $\text{Ext}_\Lambda^i(T, \Lambda)$ with T in $\text{mod } \Lambda$ and $i \leq k$, we have $\text{grade } X \geq i$. Then we have $\text{grade } \text{Ext}_\Lambda^j(T, \Lambda) \geq j$ for $j = 1, 2$. By (1) every composition factor S of $\text{Ext}_\Lambda^1(T, \Lambda)$ has grade at least 1, and hence 2 by (2). Then by an induction on the length of $\text{Ext}_\Lambda^1(T, \Lambda)$, we get $\text{grade } \text{Ext}_\Lambda^1(T, \Lambda) \geq 2$.

(4) is a direct result of (1) and (3). \square

In the following we recall some basic properties of τ -rigid modules. We start with the following definition [AIR].

Definition 2.4. We call $M \in \text{mod } \Lambda$ τ -rigid if $\text{Hom}_\Lambda(M, \tau M) = 0$. In addition, M is called τ -tilting if M is τ -rigid and $|M| = |\Lambda|$. Moreover, M is called *support τ -tilting* if there exists an idempotent e of Λ such that M is a τ -tilting $\Lambda/(e)$ -module.

It is clear that any τ -rigid Λ -module M is rigid, that is, $\text{Ext}_\Lambda^1(M, M) = 0$. In general the converse is not true. But if $\text{pd}_\Lambda M = 1$, then M is τ -rigid if and only if M is rigid. Recall that a Λ -module T is called a (*classical*) *tilting module* if T satisfies (1) $\text{pd}_\Lambda T \leq 1$, (2) $\text{Ext}_\Lambda^1(T, T) = 0$ and (3) $|T| = |\Lambda|$. It is showed in [AIR] that a tilting Λ -module is exactly a faithful support τ -tilting Λ -module.

To judge τ -rigid modules of projective dimension 2 over Auslander algebras, we also need the following lemma in [AIR].

Lemma 2.5. *Let Λ be an algebra and M a Λ -module without projective direct summands. Then M is τ -rigid in $\text{mod } \Lambda$ if and only if $\text{Tr } M$ is τ -rigid in $\text{mod } \Lambda^{\text{op}}$.*

Recall that a morphism $f : M \rightarrow N$ is called *right minimal* (resp. *left minimal*) if $fg = f$ (resp. $gf = f$) implies that g is an isomorphism, where g is a homomorphism of the form $M \rightarrow M$ (resp. $N \rightarrow N$). The following properties of right minimal (resp. left minimal) morphisms in [HuZ] are useful for the proof of the main results.

Lemma 2.6. *Let $0 \rightarrow A \xrightarrow{g} B \xrightarrow{f} C \rightarrow 0$ be a non-split exact sequence in $\text{mod } \Lambda$ with B projective-injective. Then the following are equivalent:*

- (1) *A is indecomposable and g is left minimal.*
- (2) *C is indecomposable and f is right minimal.*

3. MAIN RESULTS

In this section we give the main results of this paper and some examples to show the main results. Throughout this section, $\Lambda = \text{End}_R A$ is the Auslander algebra of a representation-finite algebra R with an additive generator A .

It is showed by Igusa [Ig] that S is rigid for any simple module S over an algebra Γ of finite global dimension. However, we give a new direct proof for the rigidity of simple modules whenever Γ is an Auslander algebra.

Proposition 3.1. *Let Λ be an Auslander algebra and S a simple Λ -module. Then $\text{Ext}_\Lambda^1(S, S) = 0$.*

Proof. For a simple Λ -module S , we show the assertion by using the projective dimension of S .

If $\text{pd}_\Lambda S = 0$, there is nothing to show.

If $\text{pd}_\Lambda S = 1$, then we can get a minimal projective resolution $0 \rightarrow P_1(S) \rightarrow P_0(S) \rightarrow S \rightarrow 0$. Then the length of $P_1(S)$ is smaller than that of $P_0(S)$, and hence $\text{Hom}_\Lambda(P_1(S), S) = 0$. So one gets $\text{Ext}_\Lambda^1(S, S) \simeq \text{Hom}_\Lambda(P_1(S), S) = 0$.

If $\text{pd}_\Lambda S = 2$, then by Proposition 2.2, there is an AR -sequence $0 \rightarrow \tau X \rightarrow E \rightarrow X \rightarrow 0$ in $\text{mod } R$ such that $0 \rightarrow \text{Hom}_R(A, \tau X) \rightarrow \text{Hom}_R(A, E) \rightarrow \text{Hom}_R(A, X) \rightarrow S \rightarrow 0$ is a minimal projective

resolution of S . On the contrary, suppose that $\text{Ext}_\Lambda^1(S, S) \neq 0$, then we get that $\text{Hom}_\Lambda(P_1(S), S) \simeq \text{Ext}_\Lambda^1(S, S) \neq 0$. So $P_0(S) = \text{Hom}_R(A, X)$ is a direct summand of $P_1(S) = \text{Hom}_\Lambda(A, E)$. Note that the functor $\text{Hom}_\Lambda(A, -)$ induces an equivalence from $\text{add} A$ to $\text{add} \Lambda$, then X is a direct summand of E . Since $E \rightarrow X$ is right almost split, then we get an irreducible morphism $f : X \rightarrow X$ by [AsSS, IV, Theorem 1.10(b)], a contradiction. \square

Denote by $(-)^*$ the functor $\text{Hom}_\Lambda(-, \Lambda)$, then we have the following lemma [IZ] with a different shorter proof.

Lemma 3.2. *Let Λ be an Auslander algebra, and let M be a Λ -module with $\text{pd}_\Lambda M \leq 1$. Then the canonical map $M \xrightarrow{\varphi_M} M^{**}$ is injective, and the projective dimension of any composition factor of M^{**}/M is 2.*

Proof. By [AuB], we get an exact sequence $0 \rightarrow \text{Ext}_{\Lambda^{op}}^1(\text{Tr} M, \Lambda) \rightarrow M \rightarrow M^{**} \rightarrow \text{Ext}_{\Lambda^{op}}^2(\text{Tr} M, \Lambda) \rightarrow 0$. To show the former assertion, it suffices to show that $\text{Ext}_{\Lambda^{op}}^1(\text{Tr} M, \Lambda) = 0$. In the following we show $\text{grade Tr } M = 2$. Since $\text{pd}_\Lambda M \leq 1$, then one gets $\text{Tr} M \simeq \text{Ext}_\Lambda^1(M, \Lambda)$ and hence by Lemma 2.3(3), $\text{grade Tr } M \geq 2$ holds, and hence $\text{Ext}_{\Lambda^{op}}^1(\text{Tr} M, \Lambda) = 0$. We get the desired injection. Then by Lemma 2.3(2), the later assertion holds. \square

Now we are in a position to state the following main result on the (τ) -rigidness of modules with projective dimension 1.

Theorem 3.3. *Let Λ be an Auslander algebra and M a Λ -module with $\text{pd}_\Lambda M = 1$. Then $\text{Ext}_\Lambda^1(M, M) = 0$ if and only if $\text{Ext}_\Lambda^2(N, M) = 0$ holds for $N = M^{**}/M$.*

Proof. We show the assertion step by step.

(1) For any $M \in \text{mod } \Lambda$, M^* is projective. Here we only need the condition $\text{gl.dim } \Lambda = 2$.

Let $P_1(M) \rightarrow P_0(M) \rightarrow M \rightarrow 0$ be a projective resolution of M . Applying the functor $(-)^*$, we get an exact sequence $0 \rightarrow M^* \rightarrow P_0(M)^* \rightarrow P_1(M)^*$. Since $\text{gl.dim } \Lambda \leq 2$, one gets that M^* is a projective Λ^{op} -module. Thus M^{**} is a projective Λ -module.

(2) $\text{Ext}_\Lambda^1(M, M) \simeq \text{Ext}_\Lambda^2(M^{**}/M, M)$ holds.

By Lemma 3.2, we get the exact sequence $0 \rightarrow M \rightarrow M^{**} \rightarrow \text{Ext}_{\Lambda^{op}}^2(\text{Tr} M, \Lambda) (= M^{**}/M) \rightarrow 0$. Applying the functor $\text{Hom}_\Lambda(-, M)$ to the exact sequence, we get the desired isomorphism since M^{**} is projective by (1). \square

Immediately, we have the following corollary.

Corollary 3.4. *Let Λ be an Auslander algebra and M a Λ -module with $\text{pd}_\Lambda M = 1$.*

- (1) *If $\text{id}_\Lambda M = 1$, then $\text{Ext}_\Lambda^1(M, M) = 0$ holds.*
- (2) *If $\text{Ext}_\Lambda^2(S', M) = 0$ holds for any composition factor S' of M^{**}/M , then $\text{Ext}_\Lambda^1(M, M) = 0$ holds.*

Proof. (1) follows from Theorem 3.3 directly. By induction on the length of M^{**}/M , one can get the assertion (2). \square

Remark 3.5. We should remark that the converse of Corollary 3.4 are not true in general (see Example 3.11(5)).

Denote by $\text{ir-rig } \Lambda$ the set of isomorphism classes of indecomposable τ -rigid Λ -modules. Similarly, one can define $\text{ir-rig } \Lambda^{op}$. Denote by \mathcal{G} the subset of $\text{ir-rig } \Lambda$ consisting of isomorphism classes of τ -rigid modules of grade 2 and denote by \mathcal{S} the subset of $\text{ir-rig } \Lambda^{op}$ consisting of isomorphism classes of non-projective τ -rigid submodules of $\text{add } \Lambda^{op}$. To judge τ -rigid modules of projective dimension 2 over Auslander algebras, we need the following proposition.

Proposition 3.6. *Let Λ be an algebra of global dimension 2. There is a bijection between \mathcal{G} and \mathcal{S} via $\text{Tr} : M \mapsto \text{Tr} M$.*

Proof. By Lemma 2.5 M is τ -rigid if and only if $\text{Tr } M$ is τ -rigid. Now it suffices to show that (a) $M \in \mathcal{G}$ implies that $\text{Tr } M \in \mathcal{S}$ and (b) $M \in \mathcal{S}$ implies that $\text{Tr } M \in \mathcal{G}$.

(a) Since $M \in \mathcal{G}$, take the following minimal projective resolution of M : $\cdots \rightarrow P_1(M) \rightarrow P_0(M) \rightarrow M \rightarrow 0$. Applying the functor $(-)^*$, we get an exact sequence

$$0 = M^* \rightarrow P_0(M)^* \rightarrow P_1(M)^* \rightarrow \text{Tr } M \rightarrow 0, \quad (3.1)$$

which is a minimal projective resolution of $\text{Tr } M$. Then $\text{pd}_\Lambda \text{Tr } M = 1$. On the other hand, since $\text{grade } M = 2$, one gets the following sequences

$$0 = M^* \rightarrow P_0(M)^* \rightarrow \Omega^1 M^* \rightarrow \text{Ext}_\Lambda^1(M, \Lambda) = 0. \quad (3.2)$$

and

$$0 \rightarrow \Omega^1 M^* \rightarrow P_1(M)^* \rightarrow P_2(M)^* \quad (3.3)$$

Comparing exact sequences 3.1 with 3.2 and 3.3, one gets that $\text{Tr } M$ is a submodule of $P_2(M)^*$.

(b) Since $M \in \mathcal{S}$ is non-projective and $\text{gl.dim } \Lambda = 2$, then $\text{pd}_\Lambda M = 1$. Take a minimal projective resolution of M : $0 \rightarrow P_1(M) \rightarrow P_0(M) \rightarrow M \rightarrow 0$. Applying $(-)^*$, we get the following exact sequence $0 \rightarrow M^* \rightarrow P_0(M)^* \rightarrow P_1(M)^* \rightarrow \text{Tr } M \rightarrow 0$. Note that Tr is a duality and $\text{pd}_\Lambda M = 1$, one gets that $\text{Hom}_{\Lambda^{op}}(\text{Tr } M, \Lambda) = 0$. Since M can be embedded into a projective module, then M is torsionless, that is $M \rightarrow M^{**}$ is injective. By [AuB] there is an exact sequence $0 \rightarrow \text{Ext}_{\Lambda^{op}}^1(\text{Tr } M, \Lambda) \rightarrow M \rightarrow M^{**} \rightarrow \text{Ext}_{\Lambda^{op}}^2(\text{Tr } M, \Lambda) \rightarrow 0$ which implies that $\text{Ext}_{\Lambda^{op}}^1(\text{Tr } M, \Lambda) = 0$. Then $\text{grade } \text{Tr } M = 2$. \square

As a corollary, we get the following.

Corollary 3.7. *Let Λ be an Auslander algebra and $M \in \text{mod } \Lambda$. If M is of grade 2, then M is τ -rigid if and only if $\text{Tr } M$ is τ -rigid with $\text{pd}_\Lambda \text{Tr } M = 1$ in $\text{mod } \Lambda^{op}$.*

Proof. By Proposition 3.6, it is enough to show that $\text{pd}_\Lambda M = 1$ if and only if M can be embedded into a projective module. Since $\text{gl.dim } \Lambda = 2$, one gets that M can be embedded into a projective module implies that $\text{pd}_\Lambda M = 1$. The converse follows from Lemma 3.2. \square

Recall that from [DIJ] that an algebra Λ is called τ -tilting finite if there are finite number of non-isomorphic indecomposable τ -rigid modules in $\text{mod } \Lambda$. It is clear that a τ -tilting finite algebra admits finite number of tilting Λ -modules and tilting Λ^{op} -modules. To find a way from two-sided tilting finite to τ -tilting finite, we have the following.

Theorem 3.8. *Let Λ be an algebra of global dimension 2 admitting finite number of basic tilting Λ -modules and tilting Λ^{op} -modules. If all indecomposable τ -rigid modules M with $\text{pd}_\Lambda M = 2$ are of grade 2, then Λ is τ -tilting finite.*

Proof. By the assumption, there are finite number of tilting modules which implies that there are finite number of indecomposable τ -rigid Λ -modules and Λ^{op} -modules of projective dimension less than or equal to 1. Then by Proposition 3.6, the number of indecomposable τ -rigid Λ -module of grade 2 is equal to the number of indecomposable non-projective τ -rigid submodules N of Λ^{op} . Since $\text{gl.dim } \Lambda = 2$, we get that $\text{pd}_\Lambda N = 1$, and hence the number of this class of modules is finite. Note that all indecomposable τ -rigid Λ -modules with projective dimension 2 are of grade 2, then the number of indecomposable τ -rigid modules with projective dimension 2 is finite by Proposition 3.6. \square

Immediately, we have the following corollary which confirms the τ -tilting finiteness of the the Auslander algebra of $K[x]/(x^n)$ showed in [IZ].

Corollary 3.9. *Let Λ be an Auslander algebra admitting finite number of basic tilting Λ -modules and tilting Λ^{op} -modules. If all indecomposable τ -rigid modules M with $\text{pd}_\Lambda M = 2$ are of grade 2, then Λ is τ -tilting finite.*

For a module M , denote by $\text{rad}M$ and $\text{soc}M$ the radical and the socle of M , respectively. Now we give the following classification of Auslander algebras admitting a unique simple module of projective dimension 2 which gives a support to Theorem 3.3 and Corollary 3.9.

Theorem 3.10. *Let Λ be an Auslander algebra. If Λ admits a unique simple Λ -module S with $\text{pd}_\Lambda S = 2$, then*

- (1) *Λ is either the Auslander algebra of the path algebra $R = KQ$ with $Q : 1 \rightarrow 2$ or the Auslander algebra of the Nakayama local algebra R of radical square zero.*
- (2) *Every indecomposable Λ -module M with $\text{pd}_\Lambda M \leq 1$ is rigid, and hence τ -rigid.*
- (3) *All indecomposable τ -rigid Λ -modules N with $\text{pd}_\Lambda N = 2$ are of grade 2.*

Proof. Since (2) and (3) follow from (1) easily, we only show (1). By Proposition 2.2, there is a unique non-projective indecomposable R -module X such that the AR -sequence $0 \rightarrow \tau X \rightarrow E \rightarrow X \rightarrow 0$ in $\text{mod}R$ induces a minimal projective resolution of S : $0 \rightarrow \text{Hom}_R(A, \tau X) \rightarrow \text{Hom}_R(A, E) \rightarrow \text{Hom}_R(A, X) \rightarrow S \rightarrow 0$. Then all indecomposable modules are projective except X . We claim that X should be simple. Otherwise, there would be a simple factor module Y of X such that $Y \not\cong X$. By the proof above Y would be projective and hence $X \simeq Y$ is projective, a contradiction. Now we divide the proof in two parts.

(a) If X is not injective, then all indecomposable injective R -modules are projective, and hence R is self-injective. So we get that R is local with a unique simple module X . Otherwise, there would be a simple projective-injective R -module. One gets a contradiction since R is basic and connected. Taking a minimal projective resolution of X , we get the following exact sequence $0 \rightarrow \Omega^1 X \rightarrow P_0(X) (= R) \rightarrow X \rightarrow 0$. By Lemma 2.6, $\Omega^1 X$ is indecomposable non-projective, and hence $\Omega^1 X \simeq X$. Then $\text{rad}^2 R = 0$ holds. By [AuRS, IV, Proposition 2.16], R is a Nakayama algebra.

(b) If X is injective, then $X \not\cong \text{soc}P$ for any indecomposable projective R -module. Hence the injective envelope $I^0(R)$ is projective, that is, R is Auslander's 1-Gorenstein [FGR]. Then $P_0(X)$ is projective-injective since X is injective. Taking a part of minimal projective resolution of X : $0 \rightarrow \Omega^1 X \rightarrow P_0(X) \rightarrow X \rightarrow 0$, one gets that $\Omega^1 X$ is indecomposable and projective by Lemma 2.6. Then we conclude that R is a hereditary algebra.

In the following we show R is a Nakayama algebra. One can show that $P_0(X)$ is the unique projective-injective module in $\text{mod}R$ since R is a basic connected hereditary algebra. Then every indecomposable projective R -module is contained in $P_0(X)$ and admits a unique composition series. By [FGR], R^{op} is also Auslander's 1-Gorenstein. Similarly, every indecomposable projective R^{op} -module admits a unique composition series. So R is a Nakayama algebra. By [AsSS, V, Theorem 3.2] and the fact all indecomposable R -modules are projective except one, we get that $R = KQ$ with $Q : 1 \rightarrow 2$. \square

At the end of this paper we give another two examples to show our main results.

Example 3.11. Let Λ be the Auslander algebra of $K[x]/(x^n)$. Then we have the following:

- (1) Λ is given by

$$1 \begin{array}{c} \xrightarrow{a_1} \\ \xleftarrow{b_2} \end{array} 2 \begin{array}{c} \xrightarrow{a_2} \\ \xleftarrow{b_3} \end{array} 3 \begin{array}{c} \xrightarrow{a_3} \\ \xleftarrow{b_4} \end{array} \cdots \begin{array}{c} \xrightarrow{a_{n-2}} \\ \xleftarrow{b_{n-1}} \end{array} n-1 \begin{array}{c} \xrightarrow{a_{n-1}} \\ \xleftarrow{b_n} \end{array} n$$

with relations $a_1 b_2 = 0$ and $a_i b_{i+1} = b_i a_{i-1}$ for any $2 \leq i \leq n-1$. Λ is of infinite representation type if $n \geq 5$.

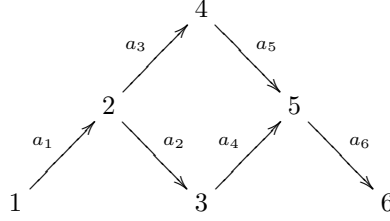
- (2) All indecomposable module M with $\text{pd}_\Lambda M = 1 = \text{id}_\Lambda M$ are direct summands of tilting modules, and hence τ -rigid.
- (3) All indecomposable τ -rigid modules of projective dimension 2 are of grade 2. (See [IZ] for details)
- (4) The number of tilting Λ -modules (resp. Λ^{op} -modules) is $n!$ ([IZ, T]). By Theorem 3.8, Λ is τ -tilting finite.

- (5) If $n = 4$, then the indecomposable module $M = \begin{smallmatrix} 2 & 4 \\ & 3 & 4 \\ & & 4 \end{smallmatrix}$ is (τ) -rigid with $\text{pd}_\Lambda M = 1$ and $\text{id}_\Lambda M = 2$ and $M^{**} = \begin{smallmatrix} 1 & 2 & 3 \\ & 2 & 4 \\ & & 3 & 4 \end{smallmatrix}$. But $\text{Ext}_\Lambda^2(S(2), M) \neq 0$.

We should remark that there does exist an Auslander algebra Λ such that an indecomposable τ -rigid Λ -module with projective dimension 2 does not necessarily have grade 2.

Example 3.12. Let Λ be the Auslander algebra of KQ with $Q : 1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3$. Then

- (1) Λ is given by the following quiver Q' :



with relations $a_2a_1 = 0$, $a_5a_3 = a_4a_2$ and $a_6a_4 = 0$.

- (2) All indecomposable modules are τ -rigid.
 (3) The indecomposable module $M = \begin{smallmatrix} 2 & 4 \\ & 3 & 4 \end{smallmatrix}$ is of projective dimension 2, but it is not of grade 2 since $\text{pd}_\Lambda S(4) = 1$.

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